Chapter 2: Systems of Ordinary Differential Equations

Perla Mallouk

perla.mallouk@math.cnrs.fr

MAP5, Université Paris Cité

Systems of ODEs – Definition

A system of ODEs is a system of several differential equations involving several unknown functions $x(t)$, $y(t)$, $z(t)$, etc.

The **dimension** of the system corresponds to the number of unknown functions.

For example,

$$\begin{cases} x'' + xy = z \\ y'z = \cos(tx) \\ z'' = e^{xy} \end{cases}$$

is a system of ODEs of **dimension 3** and **order 2** (since derivatives up to order 2 are involved).

Systems of ODEs – Transformation to first order

Any ODE can be transformed into a **system of ODEs of order 1**.
$\rightarrow$ It is practical if we have ODEs of order higher than 1.

Example:

- The following ODE $\log(y'') - y'^2 + 3y = t$ (order 2) is equivalent to the order 1 system

$$\begin{cases} z = y' \\ \log(z') - z^2 + 3y = t \end{cases}$$

Systems of ODEs – Matrix form

Any **first-order explicit linear system of ODEs with constant coefficients** can be written in matrix form.

Example:

$$\begin{cases} x' = 3x - z + e^t \\ y' = -3x - t \\ z' = x + y + z \end{cases} \quad \Leftrightarrow \quad X' = AX + B(t)$$

Systems of ODEs – Matrix form

Any **first-order explicit linear system of ODEs with constant coefficients** can be written in matrix form.

Example:

$$\begin{cases} x' = 3x - z + e^t \\ y' = -3x - t \\ z' = x + y + z \end{cases} \quad \Leftrightarrow \quad X' = AX + B(t)$$

with

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad A = \begin{pmatrix} 3 & 0 & -1 \\ -3 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad \text{and} \quad B(t) = \begin{pmatrix} e^t \\ -t \\ 0 \end{pmatrix}.$$

Systems of ODEs – Matrix form

We are left to study first-order ODEs of the form $X'(t) = AX(t) + B(t)$ where:

- $X(t)$ is an unknown vector-valued function of size n,
- $B(t)$ is a known vector-valued function of size n,
- A is a known $n \times n$ matrix with constant coefficients.

We will see how to solve this system when A is **diagonalizable** and when the system is **homogeneous** ($B(t) = 0$) or when $B(t)$ is a **constant vector** ($B(t) = B$ for all t).

Preliminaries: eigenvalues and eigenvectors

Definition

Let A be an $n \times n$ matrix.
A non-zero vector $V \in \mathbb{R}^n$ is called an **eigenvector** of A if there exists a scalar $\lambda \in \mathbb{R}$ such that:

$$AV = \lambda V.$$

The scalar λ is called an **eigenvalue** of A corresponding to the eigenvector V.

Remark: We denote $\mathrm{Sp}(A)$ the set of eigenvalues of A.

Preliminaries: eigenvalues and eigenvectors

Let A be an $n \times n$ matrix.

1. First, **find the eigenvalues** λ of A by solving the equation $\det(A - \lambda I) = 0$.
2. Then, for each eigenvalue λ, **find the associated eigenvectors** $V \neq 0$ by solving the system $(A - \lambda I)V = 0$.

Remark: You can verify your work by making sure that $AV = \lambda V$ for each eigenvalue λ and its associated eigenvector V.

Example:

$$A = \begin{pmatrix} 5 & -2 \\ -1 & 4 \end{pmatrix}$$

Preliminaries: eigenvalues and eigenvectors

Let A be an $n \times n$ matrix.

1. First, **find the eigenvalues** λ of A by solving the equation $\det(A - \lambda I) = 0$.
2. Then, for each eigenvalue λ, **find the associated eigenvectors** $V \neq 0$ by solving the system $(A - \lambda I)V = 0$.

Remark: You can verify your work by making sure that $AV = \lambda V$ for each eigenvalue λ and its associated eigenvector V.

Example:

$A = \begin{pmatrix} 5 & -2 \\ -1 & 4 \end{pmatrix}$ has eigenvalues $\lambda_1 = 3$ with eigenvector $V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\lambda_2 = 6$ with eigenvector $V_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. Here, $\mathrm{Sp}(A) = \{3, 6\}$.

Systems of ODEs – Solving techniques

$$X'(t) = AX(t)$$

Theorem

Let A be an $n \times n$ diagonalizable matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and associated eigenvectors $V_1, \ldots, V_n$. Then the general solution of $X'(t) = AX(t)$ is

$$X(t) = C_1 e^{\lambda_1 t} V_1 + \cdots + C_n e^{\lambda_n t} V_n, \qquad C_1, \ldots, C_n \in \mathbb{R}.$$

Systems of ODEs – Solving techniques

Consider the system

$$\begin{cases} x'(t) = 5x(t) - 2y(t) \\ y'(t) = -x(t) + 4y(t) \end{cases}$$

Systems of ODEs – Solving techniques

Consider the system

$$\begin{cases} x'(t) = 5x(t) - 2y(t) \\ y'(t) = -x(t) + 4y(t) \end{cases}$$

Matrix form: $X'(t) = AX(t)$ with $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $A = \begin{pmatrix} 5 & -2 \\ -1 & 4 \end{pmatrix}$.

Systems of ODEs – Solving techniques

Consider the system

$$\begin{cases} x'(t) = 5x(t) - 2y(t) \\ y'(t) = -x(t) + 4y(t) \end{cases}$$

Matrix form: $X'(t) = AX(t)$ with $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $A = \begin{pmatrix} 5 & -2 \\ -1 & 4 \end{pmatrix}$.

Eigenvalues and eigenvectors: $\lambda_1 = 3$, $V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\lambda_2 = 6$, $V_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

Systems of ODEs – Solving techniques

Consider the system

$$\begin{cases} x'(t) = 5x(t) - 2y(t) \\ y'(t) = -x(t) + 4y(t) \end{cases}$$

Matrix form: $X'(t) = AX(t)$ with $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $A = \begin{pmatrix} 5 & -2 \\ -1 & 4 \end{pmatrix}$.

Eigenvalues and eigenvectors: $\lambda_1 = 3$, $V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\lambda_2 = 6$, $V_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

General solution:

$$X(t) = C_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{6t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \qquad C_1, C_2 \in \mathbb{R},$$

i.e.,

$$\begin{cases} x(t) = C_1 e^{3t} - 2C_2 e^{6t} \\ y(t) = C_1 e^{3t} + C_2 e^{6t} \end{cases}$$

Systems of ODEs – Solving techniques

As for scalar ODEs, an initial condition $X(0) = X_0$ picks out a **unique** solution among the family found above.

Example: continuing with the previous example

$$\begin{cases} x(t) = C_1 e^{3t} - 2C_2 e^{6t} \\ y(t) = C_1 e^{3t} + C_2 e^{6t} \end{cases}$$

and with the initial condition $X(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, using $x(0) = C_1 - 2C_2 = -1$ and $y(0) = C_1 + C_2 = 2$, we find $C_1 = 1$ and $C_2 = 1$.

So the unique solution is:

$$\begin{cases} x(t) = e^{3t} - 2e^{6t} \\ y(t) = e^{3t} + e^{6t} \end{cases}$$

Systems of ODEs – Solving techniques

$$X'(t) = AX(t) + B(t)$$

Theorem

As for scalar ODEs, the general solution of $X'(t) = AX(t) + B(t)$ *is*

$$X(t) = X_p(t) + X_h(t),$$

where X_p *is any particular solution, and* X_h *is the general solution of the associated homogeneous system* $X'(t) = AX(t)$.

Systems of ODEs – Solving techniques

Consider the system

$$\begin{cases} x'(t) = 5x(t) - 2y(t) \\ y'(t) = -x(t) + 4y(t) + 18 \end{cases}$$

Systems of ODEs – Solving techniques

Consider the system

$$\begin{cases} x'(t) = 5x(t) - 2y(t) \\ y'(t) = -x(t) + 4y(t) + 18 \end{cases}$$

Matrix form: $X = \begin{pmatrix} x \\ y \end{pmatrix}$, $A = \begin{pmatrix} 5 & -2 \\ -1 & 4 \end{pmatrix}$, and $B = \begin{pmatrix} 0 \\ 18 \end{pmatrix}$.

Systems of ODEs – Solving techniques

Consider the system

$$\begin{cases} x'(t) = 5x(t) - 2y(t) \\ y'(t) = -x(t) + 4y(t) + 18 \end{cases}$$

Matrix form: $X = \begin{pmatrix} x \\ y \end{pmatrix}$, $A = \begin{pmatrix} 5 & -2 \\ -1 & 4 \end{pmatrix}$, and $B = \begin{pmatrix} 0 \\ 18 \end{pmatrix}$.

Homogeneous solution: $X_h(t) = C_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{6t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$, $C_1, C_2 \in \mathbb{R}$.

Systems of ODEs – Solving techniques

Consider the system

$$\begin{cases} x'(t) = 5x(t) - 2y(t) \\ y'(t) = -x(t) + 4y(t) + 18 \end{cases}$$

Matrix form: $X = \begin{pmatrix} x \\ y \end{pmatrix}$, $A = \begin{pmatrix} 5 & -2 \\ -1 & 4 \end{pmatrix}$, and $B = \begin{pmatrix} 0 \\ 18 \end{pmatrix}$.

Homogeneous solution: $X_h(t) = C_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{6t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$, $C_1, C_2 \in \mathbb{R}$.

Particular solution: $X_p = -A^{-1}B = \begin{pmatrix} -2 \\ -5 \end{pmatrix}$.

Systems of ODEs – Solving techniques

Consider the system

$$\begin{cases} x'(t) = 5x(t) - 2y(t) \\ y'(t) = -x(t) + 4y(t) + 18 \end{cases}$$

Matrix form: $X = \begin{pmatrix} x \\ y \end{pmatrix}$, $A = \begin{pmatrix} 5 & -2 \\ -1 & 4 \end{pmatrix}$, and $B = \begin{pmatrix} 0 \\ 18 \end{pmatrix}$.

Homogeneous solution: $X_h(t) = C_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{6t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$, $C_1, C_2 \in \mathbb{R}$.

Particular solution: $X_p = -A^{-1}B = \begin{pmatrix} -2 \\ -5 \end{pmatrix}$.

General solution: $X(t) = X_p + X_h(t) = \begin{pmatrix} -2 \\ -5 \end{pmatrix} + C_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{6t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$,

i.e.,

$$\begin{cases} x(t) = -2 + C_1 e^{3t} - 2C_2 e^{6t} \\ y(t) = -5 + C_1 e^{3t} + C_2 e^{6t} \end{cases}$$

Systems of ODEs – Phase portraits

Definition

A **phase portrait** is a graphical representation of trajectories of a dynamical system in the **state space** (e.g., the (x, y) plane for a two-dimensional system).

For a system

$$\begin{cases} \dfrac{dx}{dt} = f(x, y), \\ \dfrac{dy}{dt} = g(x, y), \end{cases}$$

the phase portrait shows how solutions evolve as curves $(x(t), y(t))$ in the plane.

Systems of ODEs – Phase portraits

Definition

A **phase portrait** is a graphical representation of trajectories of a dynamical system in the **state space** (e.g., the (x, y) plane for a two-dimensional system).

For a system

$$\begin{cases} \dfrac{dx}{dt} = f(x, y), \\ \dfrac{dy}{dt} = g(x, y), \end{cases}$$

the phase portrait shows how solutions evolve as curves $(x(t), y(t))$ in the plane.

Key ideas:

- Each point represents a state of the system.
- Arrows indicate the direction of motion (the vector field).
- Equilibria appear as fixed points in the portrait.
- Useful to study stability, oscillations, and long-term behavior.

Systems of ODEs – Phase portraits

Consider a linear system

$$X' = AX, \quad \text{where } A \text{ is a } 2 \times 2 \text{ matrix.}$$

- Solutions depend on the eigenvalues and eigenvectors of A.
- Eigenvectors give the **principal directions** of motion.
- Eigenvalues determine the **behavior along these directions**:
  - if the eigenvalue is positive, all solutions on the corresponding direction go away from the origin.
  - if the eigenvalue is negative, all solutions on the corresponding direction go towards the origin.
- The combination of eigenvalues and eigenvectors determines the **phase portrait type** (node, saddle, spiral, center, etc.).

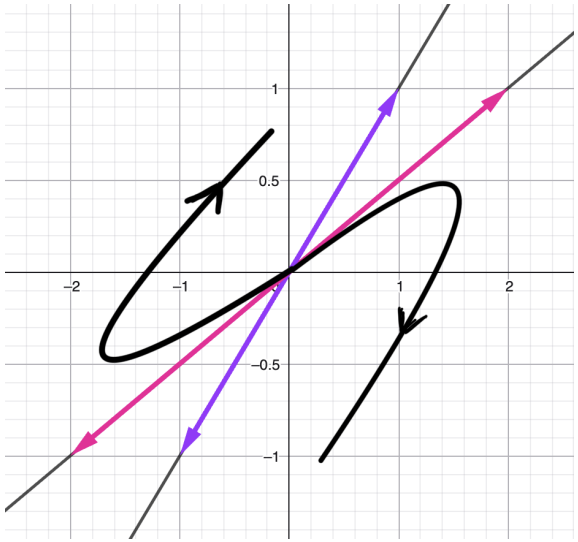
Systems of ODEs – Phase portraits

Two real eigenvalues:

$$A = \begin{pmatrix} -1 & 4 \\ -2 & 5 \end{pmatrix}$$

$$\lambda_1 = 1, \vec{v_1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 3, \vec{v_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



unstable

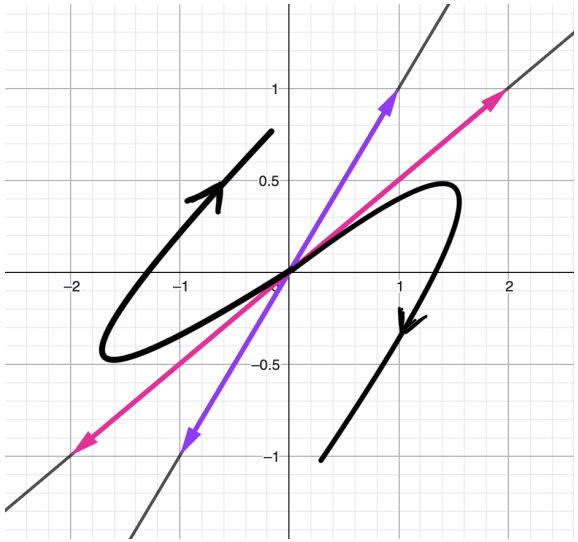
Systems of ODEs – Phase portraits

Two real eigenvalues:

$$A = \begin{pmatrix} -1 & 4 \\ -2 & 5 \end{pmatrix}$$

$$\lambda_1 = 1, \vec{v_1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 3, \vec{v_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

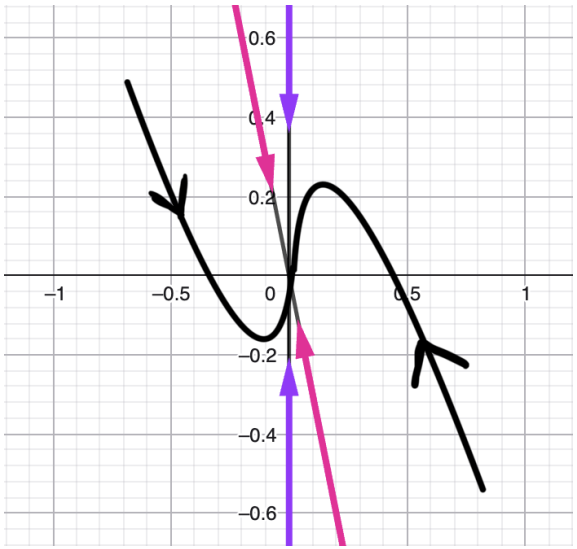


unstable

$$A = \begin{pmatrix} -3 & 0 \\ 3 & -2 \end{pmatrix}$$

$$\lambda_1 = -3, \vec{v_1} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$$\lambda_2 = -2, \vec{v_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



stable

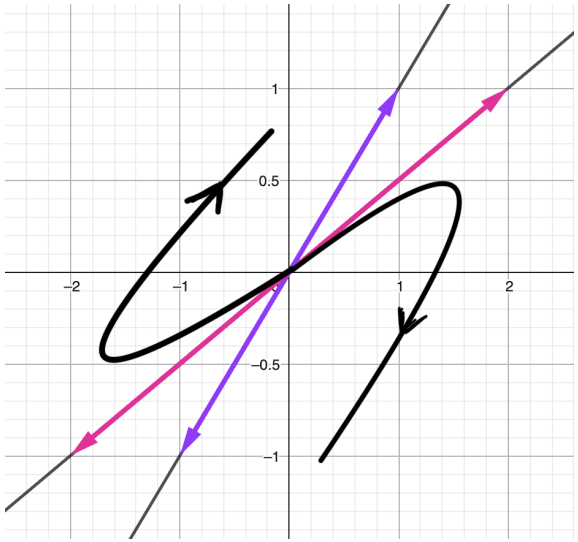
Systems of ODEs – Phase portraits

Two real eigenvalues:

$$A = \begin{pmatrix} -1 & 4 \\ -2 & 5 \end{pmatrix}$$

$$\lambda_1 = 1, \vec{v_1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 3, \vec{v_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

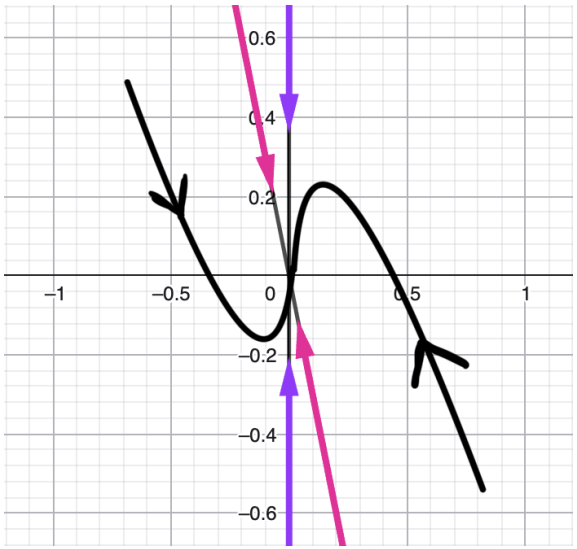


unstable

$$A = \begin{pmatrix} -3 & 0 \\ 3 & -2 \end{pmatrix}$$

$$\lambda_1 = -3, \vec{v_1} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$$\lambda_2 = -2, \vec{v_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

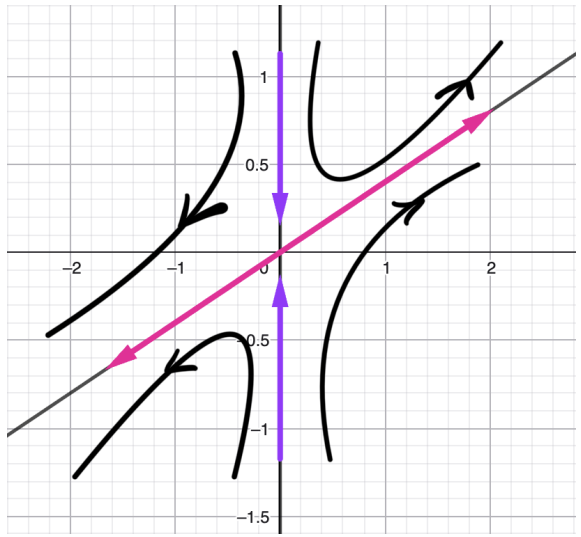


stable

$$A = \begin{pmatrix} 4 & 0 \\ 2 & -1 \end{pmatrix}$$

$$\lambda_1 = 4, \vec{v_1} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$\lambda_2 = -1, \vec{v_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



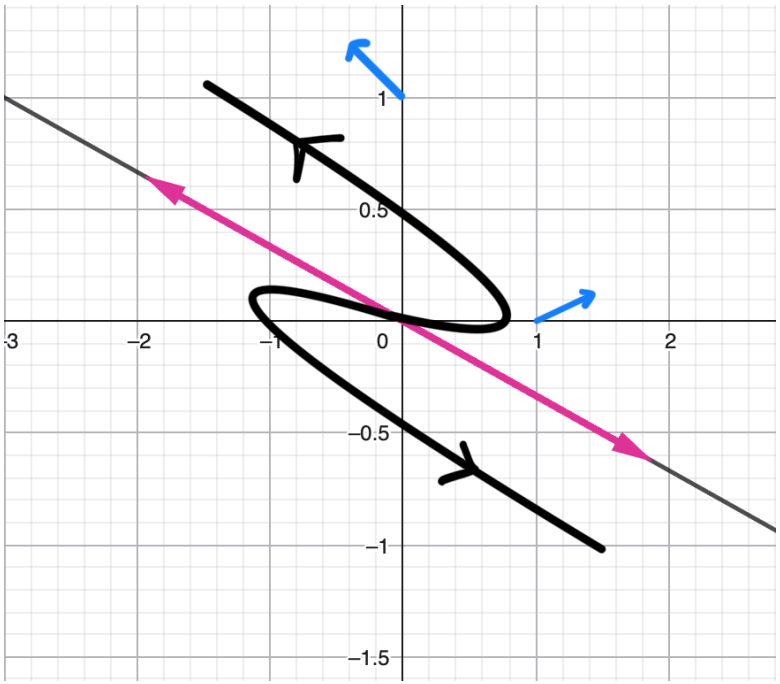
saddle

Systems of ODEs – Phase portraits

Repeated eigenvalues:

$$A = \begin{pmatrix} 2 & -3 \\ 1/3 & 4 \end{pmatrix}, \lambda = 3, \vec{v} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1/3 \end{pmatrix}, A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$$



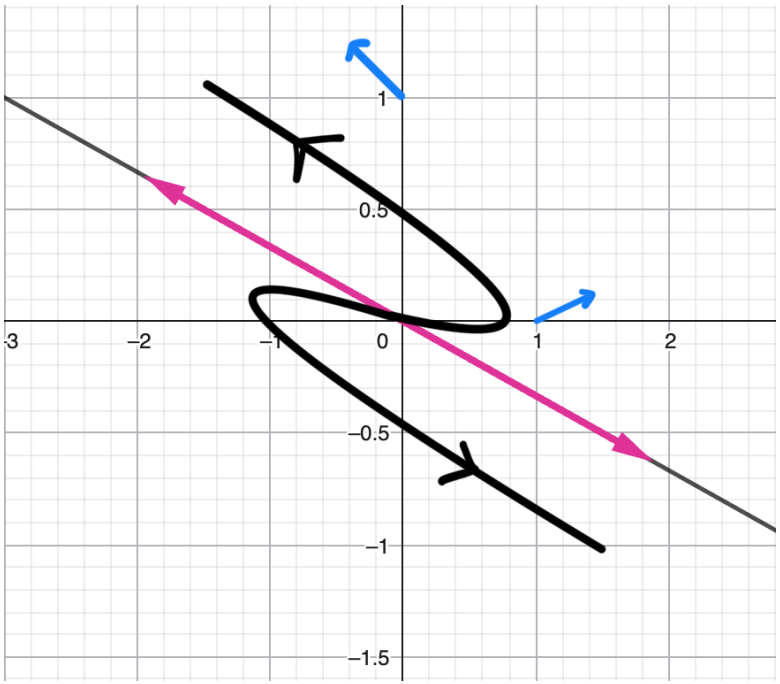
unstable

Systems of ODEs – Phase portraits

Repeated eigenvalues:

$$A = \begin{pmatrix} 2 & -3 \\ 1/3 & 4 \end{pmatrix}, \lambda = 3, \vec{v} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

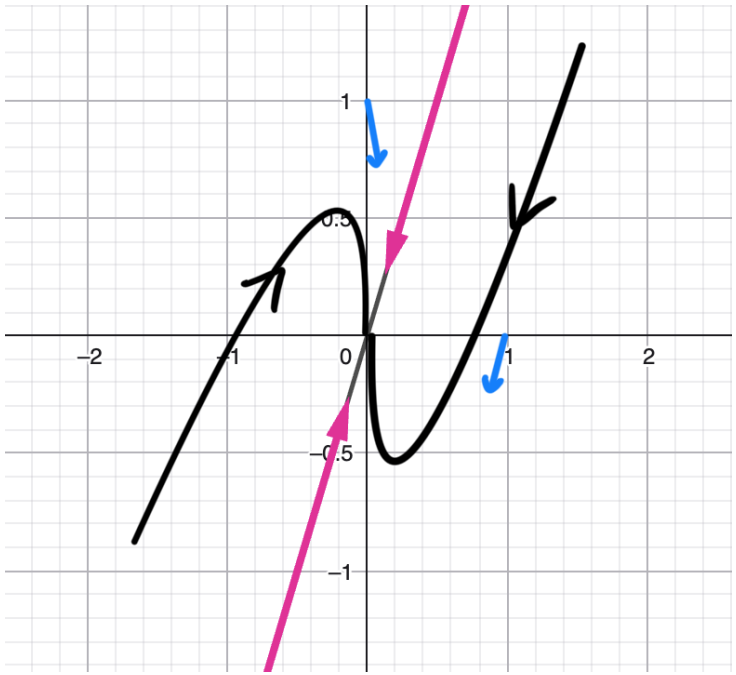
$$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1/3 \end{pmatrix}, A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$$



unstable

$$A = \begin{pmatrix} -7 & 1 \\ -4 & -3 \end{pmatrix}, \lambda = -5, \vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -7 \\ -4 \end{pmatrix}, A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$



stable

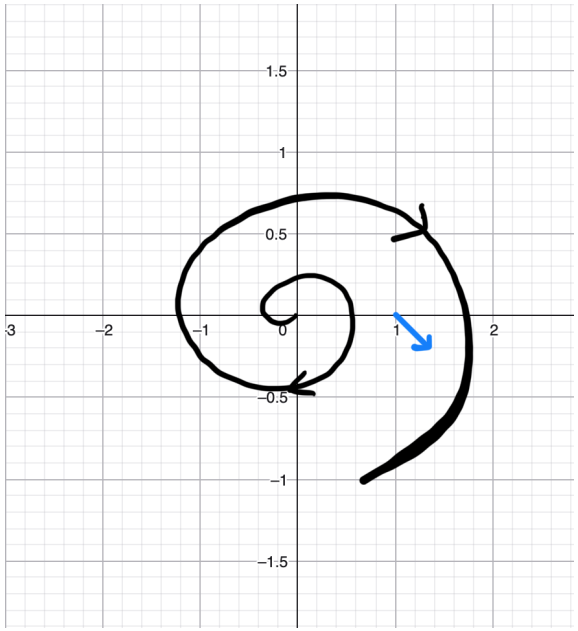
Systems of ODEs – Phase portraits

Complex eigenvalues (the real part is important):

$$A = \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}$$

$$\lambda = 2 \pm 3i$$

$$A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$



unstable spiral

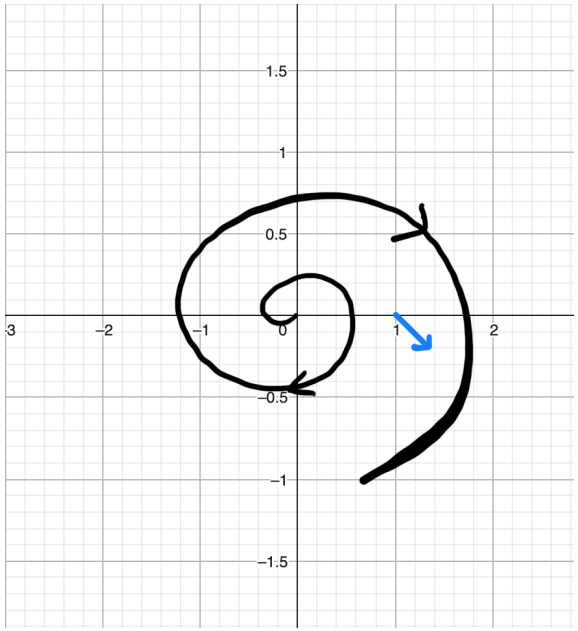
Systems of ODEs – Phase portraits

Complex eigenvalues (the real part is important):

$$A = \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}$$

$$\lambda = 2 \pm 3i$$

$$A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

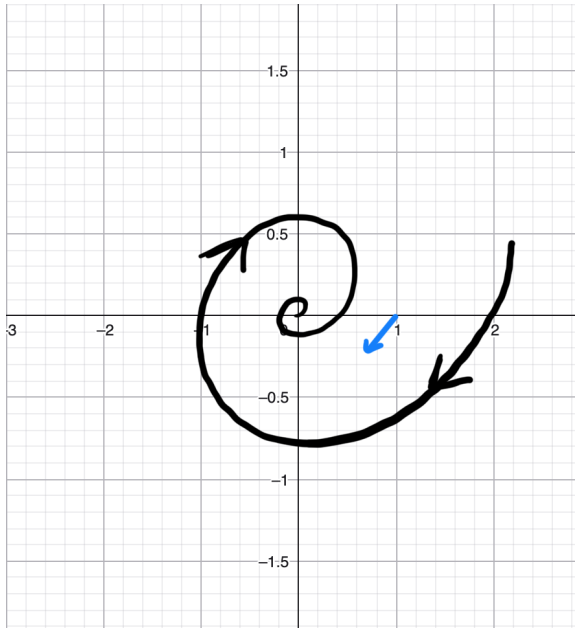


unstable spiral

$$A = \begin{pmatrix} -2 & 3 \\ -3 & -2 \end{pmatrix}$$

$$\lambda = -2 \pm 3i$$

$$A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$$



stable spiral

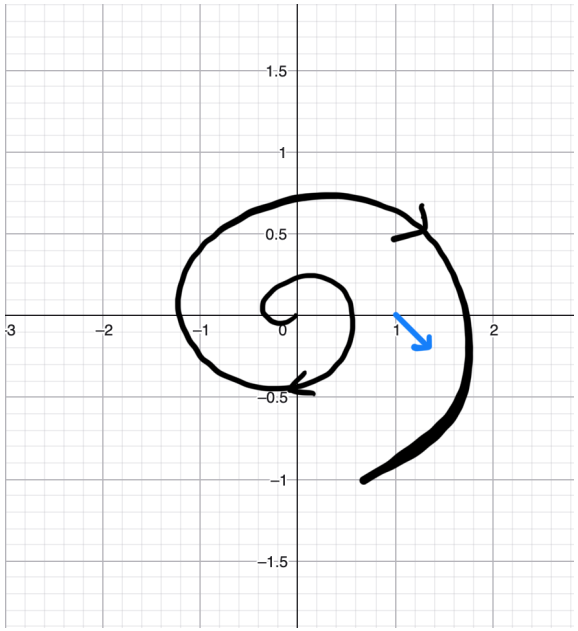
Systems of ODEs – Phase portraits

Complex eigenvalues (the real part is important):

$$A = \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}$$

$$\lambda = 2 \pm 3i$$

$$A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

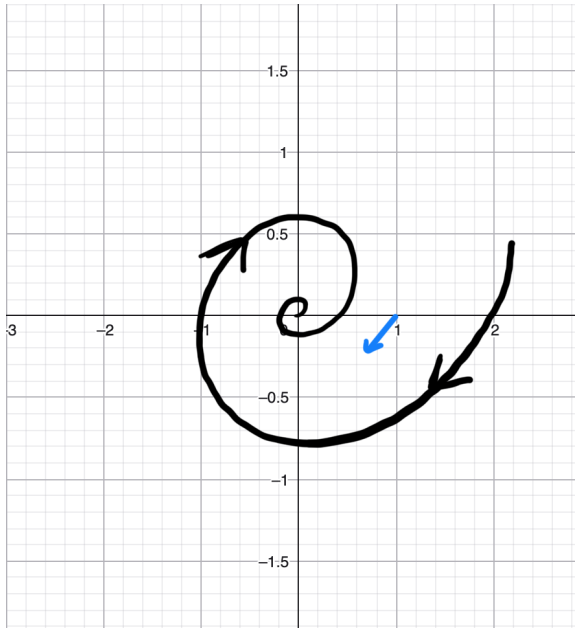


unstable spiral

$$A = \begin{pmatrix} -2 & 3 \\ -3 & -2 \end{pmatrix}$$

$$\lambda = -2 \pm 3i$$

$$A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$$

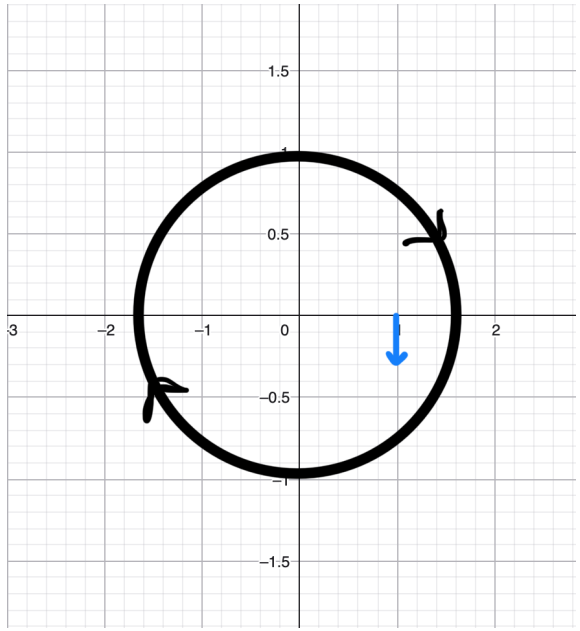


stable spiral

$$A = \begin{pmatrix} 0 & 1 \\ -5 & 0 \end{pmatrix}$$

$$\lambda = \pm\sqrt{5}i$$

$$A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -5 \end{pmatrix}$$



center

Systems of ODEs – Phase portraits

Eigenvalues	**Behavior**
Real, both positive	Unstable
Real, both negative	Stable
Real, opposite signs	Saddle
Repeated, positive	Unstable
Repeated, negative	Stable
Complex, positive real part	Unstable spiral
Complex, negative real part	Stable spiral
Complex, zero real part	Center