Chapter 2 – Exercises

Exercise 1 *(Vocabulary and matrix form)*

1. Determine the dimension and the order of the following system:

$$\begin{cases} x''(t) + \sin(y(t)) = t \\ y'(t) + x(t)^2 = z(t) \\ z'''(t) = x(t)y(t)z(t) \end{cases}$$

2. Transform the following order-3 ODE into an equivalent order-1 system:

$$y'''(t) - 2y''(t) + y'(t) - 5y(t) = e^t$$

3. Write the following system in matrix form $X'(t) = AX(t) + B(t)$:

$$\begin{cases} x'(t) = 2x(t) + y(t) - t \\ y'(t) = -x(t) + 3z(t) \\ z'(t) = 4y(t) - z(t) + e^{-t} \end{cases}$$

Exercise 2 *(Eigenvalues and eigenvectors)*

Compute the eigenvalues and associated eigenvectors of the following matrices:

1. $A_1 = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$
2. $A_2 = \begin{pmatrix} 1 & -3 \\ 3 & 7 \end{pmatrix}$
3. $A_3 = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$
4. $A_4 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 3 \\ 0 & 0 & 4 \end{pmatrix}$

Exercise 3 *(Solving homogeneous linear systems)*

Solve the following homogeneous linear systems:

1. $\begin{cases} x'(t) = 6x(t) - 2y(t) \\ y'(t) = -2x(t) + 3y(t) \end{cases}$
2. $\begin{cases} x'(t) = -4x(t) + 2y(t) \\ y'(t) = x(t) - 5y(t) \end{cases}$
3. $\begin{cases} x'(t) = 3x(t) \\ y'(t) = 4x(t) - 2y(t) \end{cases}$
4. $\begin{cases} x'(t) = 5x(t) + y(t) \\ y'(t) = 3y(t) \end{cases}$
5. $\begin{cases} x'(t) = 2x(t) + 3y(t) \\ y'(t) = x(t) + 4y(t) \\ X(0) = \begin{pmatrix} 5 \\ 1 \end{pmatrix} \end{cases}$
6. $\begin{cases} x'(t) = -2x(t) \\ y'(t) = 3x(t) + y(t) \\ X(0) = \begin{pmatrix} 4 \\ -5 \end{pmatrix} \end{cases}$

Exercise 4 *(Solving non-homogeneous linear systems)*

Solve the following non-homogeneous linear systems:

1. $\begin{cases} x'(t) = x(t) + 4y(t) + 7 \\ y'(t) = 2x(t) + 3y(t) + 4 \end{cases}$
2. $\begin{cases} x'(t) = -2x(t) - y(t) + 5 \\ y'(t) = -3x(t) - 4y(t) + 10 \end{cases}$
3. $\begin{cases} x'(t) = 4x(t) + 4 \\ y'(t) = -x(t) - 3y(t) + 5 \end{cases}$
4. $\begin{cases} x'(t) = 2y(t) - 2 \\ y'(t) = -x(t) + 3y(t) \\ X(0) = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \end{cases}$

Exercise 5 *(Phase portraits)*

For each of the following, classify the type and stability of the equilibrium at the origin (node, saddle, spiral, or center; stable or unstable):

1. $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$
2. $A = \begin{pmatrix} -5 & 2 \\ 1 & -4 \end{pmatrix}$
3. Eigenvalues $\lambda_{1,2} = -1 \pm 4i$
4. Eigenvalue $\lambda = 6$ (repeated)
5. $A = \begin{pmatrix} 0 & 7 \\ -7 & 0 \end{pmatrix}$
6. $A = \begin{pmatrix} -6 & 0 \\ 0 & 2 \end{pmatrix}$