Chapter 1: Ordinary Differential Equations (ODEs)

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Partial Differential Equation (PDE)[1]: $\partial_t y(t,x) + \partial_x y(t,x) = 0$
$\rightarrow$ the **unknown** is a **function** that depends on multiple variables.

[1] Evans, L. C. (2022). *Partial differential equations* (Vol. 19). American Mathematical Society.

ODEs

Ordinary Differential Equations (ODEs): equations involving an unknown function $y(t)$, its derivative $y'(t)$, and possibly its higher-order derivatives $y''(t)$, $y^{(3)}(t)$, *etc.*, up to a given order $y^{(n)}(t)$.

Examples:

- $y'(t) + y(t) = 0$,
- $y'(t) = ty(t) + t^2$,
- $y''(t) = \cos(y'(t))\, y(t)$.

ODEs – Notation

Some remarks about **notation**:

- Derivatives are sometimes written as $\frac{dy}{dt}(t)$, $\frac{d^2y}{dt^2}(t)$, or as $\partial_t y(t)$, $\partial_{tt} y(t)$, or as $y'(t)$, $y''(t)$, or $\dot{y}(t)$, $\ddot{y}(t)$.

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- Sometimes we drop the "(t)" in the equations to simplify notation. For example, the three previous ODEs could also be written as:
  - $y' + y = 0$,
  - $y' = ty + t^2$,
  - $y'' = \cos(y')y$.

ODEs – Vocabulary

Orders of ODEs

The **order** of an equation is the order of the highest derivative that appears. For example, if the highest derivative that appears is the first derivative, the equation is of first order; if the highest derivative that appears is the second derivative, the equation is of second order.

Examples:

- $y' + y = 0$ is an ODE of order 1,
- $y' = ty + t^2$ is an ODE of order 1,
- $y'' = \cos(y')y$ is an ODE of order 2.

ODEs – Vocabulary

Autonomous ODEs

An ODE is **autonomous** when the variable t does not appear explicitly in the equation, except through the function $y(t)$ and its derivatives.

Examples:

- $y''(t) + y'(t) + y(t) = 0$ is an autonomous ODE,
- $y''(t) = t + \cos(y(t))$ is not autonomous.

ODEs – Vocabulary

Linear ODEs

A **linear** ODE is an ODE of the form

$$a_0(t)y(t) + a_1(t)y'(t) + \cdots + a_n(t)y^{(n)}(t) = b(t),$$

where the $a_i(t)$ and $b(t)$ are known functions.

- A linear ODE is **homogeneous** if and only if $b(t) = 0$.
- A linear ODE has **constant coefficients** if the $a_i(t)$ are constants.

Examples:

- $ty + e^t y' = 2t$ is a linear ODE,
- $y'' - y' + 4y = 5\cos(t)$ is a linear ODE with constant coefficients,
- $y' - ty = 0$ is a homogeneous linear ODE.

ODEs – Initial conditions

Consider the ODE

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Initial conditions pick out one: they fix the value of the function (and its derivatives) at a starting point t_0.
$\rightarrow$ An equation of order n needs n initial conditions.

Example:
$y'(t) + y(t) = 0$ with $y(0) = 1$ has the unique solution $y(t) = e^{-t}$, but with $y(0) = 2$ has the unique solution $y(t) = 2e^{-t}$.

ODEs – Equilibrium solutions

An **equilibrium solution** (or **steady state**) of an ODE is a solution that remains constant over time.
If the ODE is of the form $\frac{dy}{dt} = f(y)$, equilibrium solutions can be found by solving $f(y) = 0$.

Examples:

- For $y'(t) = y(t) - 2$, the equilibrium is $y^* = 2$.
- For $y'(t) = y(t)\big(y(t) - 2\big)$, the equilibria are $y^* = 0$ and $y^* = 2$.

ODEs – Questions to answer

Everything we have introduced so far naturally raises the following questions.

1. **Theoretical questions:** under what condition(s) does an ODE admit solutions? Does there exist a unique solution satisfying given conditions?[2]

[2] Evans, L. C. (2022). *Hsieh, P. F., & Sibuya, Y. (2012). Basic theory of ordinary differential equations. Springer Science & Business Media.*

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[3]Polyanin, A. D., & Zaitsev, V. F. (2017). *Handbook of ordinary differential equations: exact solutions, methods, and problems. Chapman and Hall/CRC.*

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3. **Numerical resolution:** can a computer calculate numerical approximations of the solutions of an ODE? How accurate is a given method compared to the theoretical solutions?[4]

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[4] Chapter 3 of our course!

ODEs – Solving techniques

First-order homogeneous linear ODEs

Form: $y'(t) + a(t)y(t) = 0$ with a a continuous function.

Solution: Let A be an antiderivative of a. Then the set of solutions of this differential equation is the set of functions of the form

$$y_h(t) = Ce^{-A(t)}, \text{ with } C \in \mathbb{R}.$$

Unique solution: A unique solution of this ODE exists if we specify an initial condition $y(t_0) = y_0$.

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Solution: Here $a(t) = 2$, so $A(t) = 2t$. The general solution is $y_h(t) = Ce^{-2t}$. Using the initial condition $y(0) = 1$, we get $C = 1$. Therefore, the unique solution is $y_h(t) = e^{-2t}$.

First-order non-homogeneous linear ODEs

Form: $y'(t) + a(t)y(t) = b(t)$ with a and b continuous functions.

Solution: Let y_p be a solution of this differential equation and A an antiderivative of a. Then the set of solutions of this differential equation is the set of functions of the form

$$y(t) = y_p(t) + Ce^{-A(t)}, \text{ with } C \in \mathbb{R}.$$

Remark: Thus, the general form of solutions of the differential equation is:
solution = particular solution y_p + general solution,

where the general solution is the solution of the associated first-order homogeneous linear ODE:

$$y'(t) + a(t)y(t) = 0.$$

ODEs – Solving techniques

How to find particular solutions? (the case where $a(t)$ is constant)

Let the ODE be $y'(t) + ay(t) = b(t)$ where a is a constant and $b(t)$ is a function of t. We want to find a particular solution $y_p(t)$.

- If $b(t)$ is a polynomial of degree n, then $y_p(t)$ is a polynomial of degree n.
- If $b(t) = P(t)e^{rt}$ where $P(t)$ is a polynomial, then $y_p(t) = Q(t)e^{rt}$ where $Q(t)$ is a polynomial.
- If $b(t) = \alpha\cos(\omega t) + \beta\sin(\omega t)$, then $y_p(t) = \lambda\cos(\omega t) + \mu\sin(\omega t)$.
- Variation of parameters (when the above methods don't apply):
  Replace the constant C of the homogeneous solution by a function $C(t)$ and solve for $C(t)$.

ODEs – Solving techniques

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- $y'(t) - 2y(t) = \cos(t) + 2\sin(t)$

  Solution: $y(t) = -\frac{4}{5}\cos(t) - \frac{3}{5}\sin(t) + Ce^{2t}$

ODEs – Slope fields

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Definition

A **slope field** (or **direction field**) is a graphical tool to visualize the solutions of a differential equation.

For the equation

$$\frac{dy}{dt} = f(y,t),$$

slope fields show how solutions behave **without solving the ODE explicitly**.

ODEs – Slope fields

Construction:

- At each point (t, y), the slope of the solution is $f(y, t)$.
- Draw a small line segment with slope $f(y, t)$.
- Solution curves can then be sketched by following these segments.

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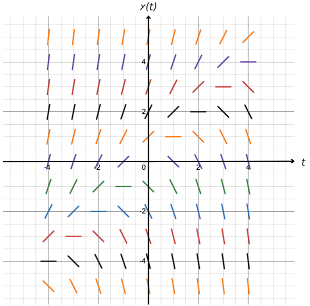

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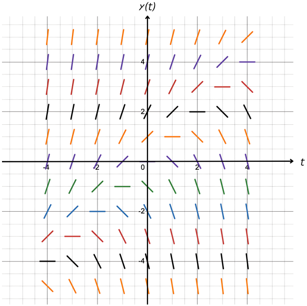

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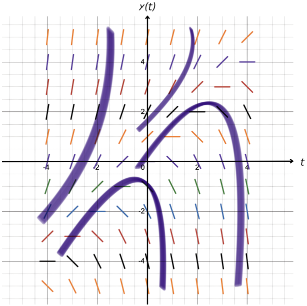

Solutions curves

ODEs – Phase diagrams

Definition

A **phase diagram** is a graphical representation of solution trajectories in the (t, y) plane for a first-order **autonomous** ODE.

For the equation

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the phase diagram shows how solutions evolve over time, depending on their initial condition.

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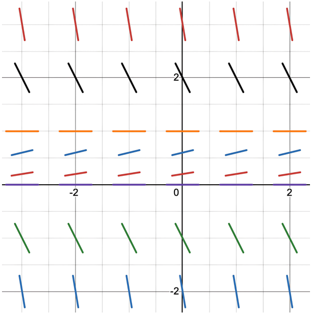
Key ideas:

- Complements slope fields: shows full solution curves, not just local slopes.
- Highlights qualitative features: growth, decay, oscillations.
- Equilibria appear as constant solutions; their **stability** can be visualized directly.

ODEs – Phase diagrams

$$\frac{dy}{dt} = y(1 - y)$$



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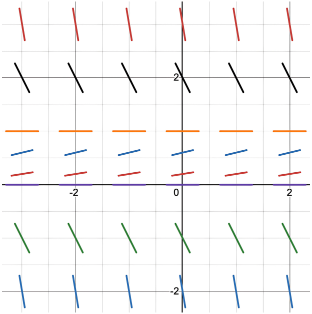

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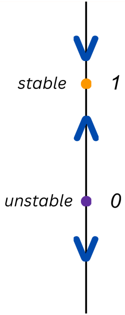

Phase diagram (solution trajectories)