Graphical solutions for ODEs and systems of ODEs

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October 14, 2025

Reminder: Lecture 3

Previously, we learned how to solve ODEs numerically (Euler's method, Runge-Kutta) given initial conditions.

Often, finding an explicit solution is impossible or impractical. In these cases, we study the **qualitative behavior** of solutions.

Graphical methods provide insight without computing the exact solution: they let us visualize how solutions evolve over time.

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In this lecture, we will explore:

- **Slope fields** and **phase diagrams** (for single ODEs)
- **Phase portraits** (for linear systems of ODEs)

Slope fields

Definition

A **slope field** (or **direction field**) is a graphical tool to visualize the solutions of a differential equation.

For the equation

$$\frac{dx}{dt} = f(x,t),$$

slope fields show how solutions behave **without solving the ODE explicitly**.

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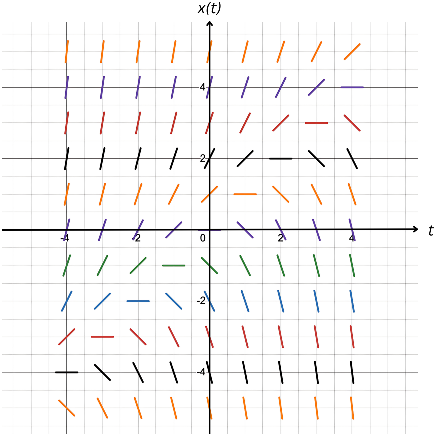
Construction:

- At each point (t,x), the slope of the solution is $f(x,t)$.
- Draw a small line segment with slope $f(x,t)$.
- Solution curves can then be sketched by following these segments.

Slope fields: example

$$\frac{dx}{dt} = x - t$$



Slope field

Slope fields: example

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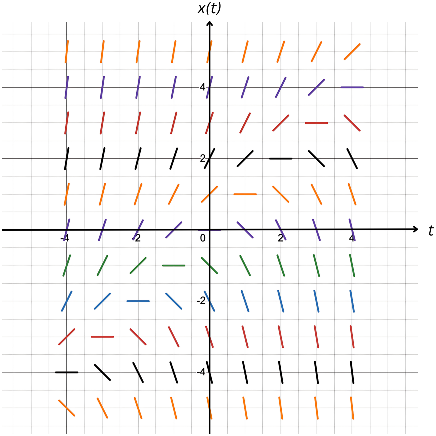
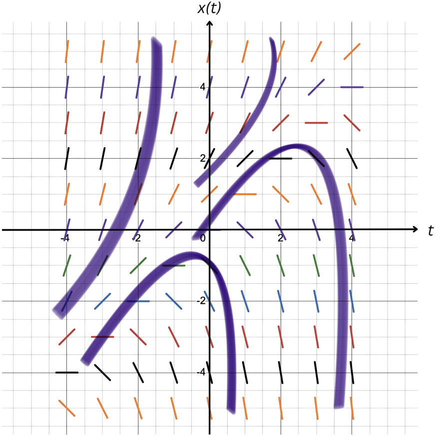


Slope field



Solutions curves

Phase diagrams

Definition

A **phase diagram** is a graphical representation of solution trajectories in the (t, x) plane for a first-order **autonomous** ODE.

For the equation

$$\frac{dx}{dt} = f(x),$$

the phase diagram shows how solutions evolve over time, depending on their initial condition.

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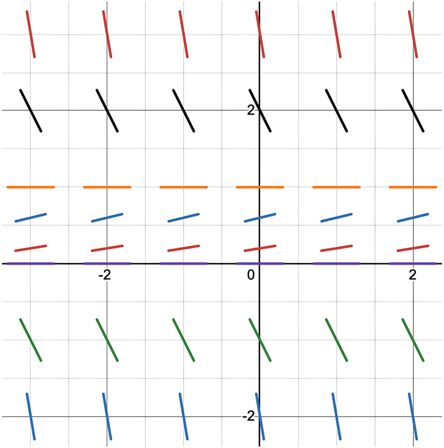
Key ideas:

- Complements slope fields: shows full solution curves, not just local slopes.
- Highlights qualitative features: growth, decay, oscillations.
- Equilibria appear as constant solutions; their **stability** can be visualized directly.

Phase diagrams: example

$$\frac{dx}{dt} = x(1 - x)$$



Slope field

Phase diagrams: example

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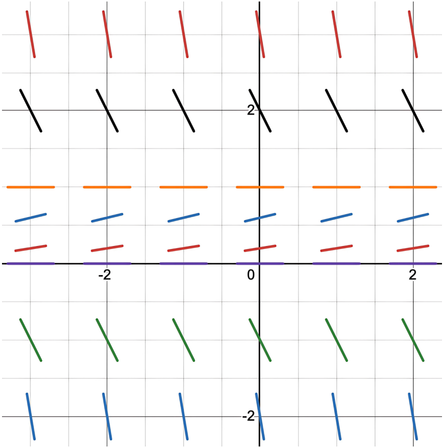
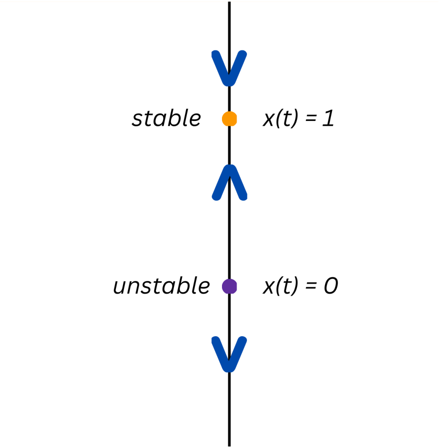


Slope field



Phase diagram (solution trajectories)

Phase portraits for linear systems of ODEs

Definition

A **phase portrait** is a graphical representation of trajectories of a dynamical system in the **state space** (e.g., the (x, y) plane for a two-dimensional system).

For a system

$$\begin{cases} \dfrac{dx}{dt} = f(x, y), \\ \dfrac{dy}{dt} = g(x, y), \end{cases}$$

the phase portrait shows how solutions evolve as curves $(x(t), y(t))$ in the plane.

Phase portraits for linear systems of ODEs

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the phase portrait shows how solutions evolve as curves $(x(t), y(t))$ in the plane.

Key ideas:

- Each point represents a state of the system.
- Arrows indicate the direction of motion (the vector field).
- Equilibria appear as fixed points in the portrait.
- Useful to study stability, oscillations, and long-term behavior.

Phase portraits for linear systems of ODEs

Consider a linear system

$$\vec{x}' = A\vec{x}, \quad \text{where } A \text{ is a } 2 \times 2 \text{ matrix.}$$

- Solutions depend on the eigenvalues and eigenvectors of A.
- Eigenvectors give the **principal directions** of motion.
- Eigenvalues determine the **behavior along these directions**:
  - if the eigenvalue is positive, all solutions on the corresponding direction go away from the origin.
  - if the eigenvalue is negative, all solutions on the corresponding direction go towards the origin.
- The combination of eigenvalues and eigenvectors determines the **phase portrait type** (node, saddle, spiral, center, etc.).

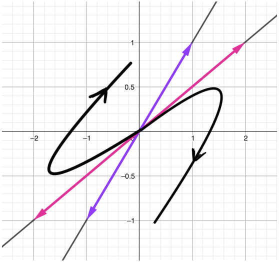
Phase portraits: examples

Two real eigenvalues:

$$A = \begin{pmatrix} -1 & 4 \\ -2 & 5 \end{pmatrix}$$

$$\lambda_1 = 1, \vec{v_1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 3, \vec{v_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



unstable

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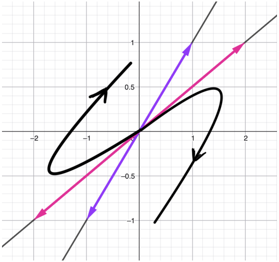

unstable

$$A = \begin{pmatrix} -3 & 0 \\ 3 & -2 \end{pmatrix}$$

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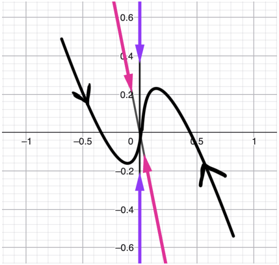

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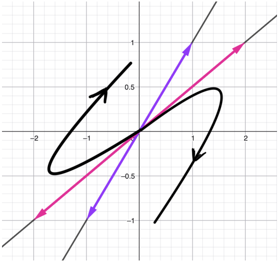

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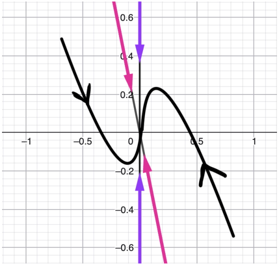
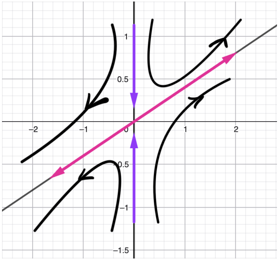


stable

$$A = \begin{pmatrix} 4 & 0 \\ 2 & -1 \end{pmatrix}$$

$$\lambda_1 = 4, \vec{v_1} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$\lambda_2 = -1, \vec{v_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



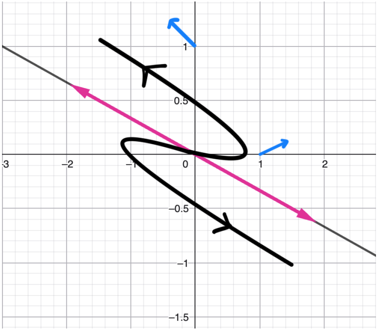
saddle

Phase portraits: examples

Repeated eigenvalues:

$$A = \begin{pmatrix} 2 & -3 \\ 1/3 & 4 \end{pmatrix}, \lambda = 3, \vec{v} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$$A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1/3 \end{pmatrix}, A \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$$



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Phase portraits: examples

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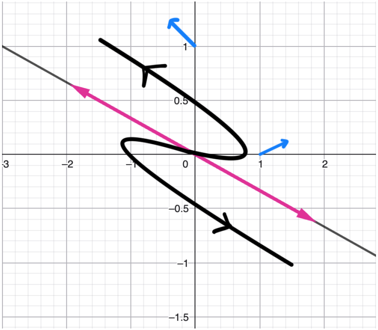

unstable

$$A = \begin{pmatrix} -7 & 1 \\ -4 & -3 \end{pmatrix}, \lambda = -5, \vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

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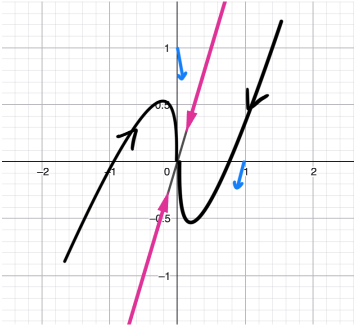

stable

Phase portraits: examples

Complex eigenvalues (the real part is important):

$$A = \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}$$

$$\lambda = 2 \pm 3i$$

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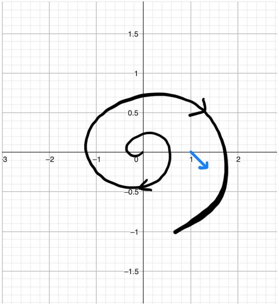

unstable spiral

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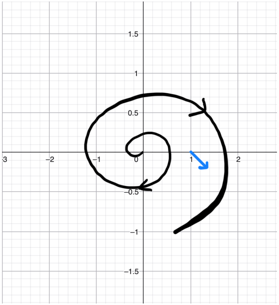

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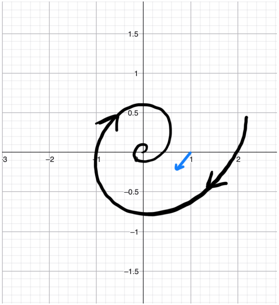

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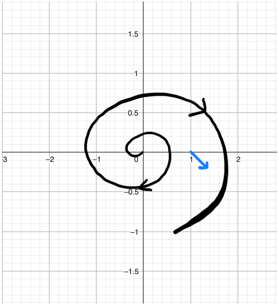

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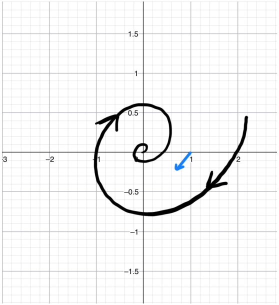
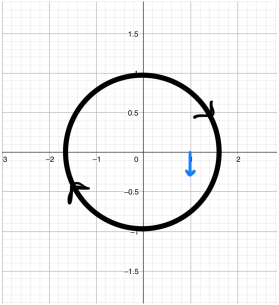


stable spiral

$$A = \begin{pmatrix} 0 & 1 \\ -5 & 0 \end{pmatrix}$$

$$\lambda = \pm\sqrt{5}i$$

$$A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -5 \end{pmatrix}$$



center

Phase portraits: summary

Eigenvalues	Behavior
Real, both positive	Unstable
Real, both negative	Stable
Real, opposite signs	Saddle
Repeated, positive	Unstable
Repeated, negative	Stable
Complex, positive real part	Unstable spiral
Complex, negative real part	Stable spiral
Complex, zero real part	Center