Systems of Ordinary Differential Equations

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Systems of ODEs

A system of ODEs is a system of several differential equations involving several unknown functions $x(t)$, $y(t)$, $z(t)$, etc.

The **dimension** of the system corresponds to the number of unknown functions.

For example,

$$\begin{cases} x'' + xy = z \\ y'z = \cos(tx) \\ z'' = e^{xy} \end{cases}$$

is a system of ODEs of **dimension 3** and **order 2** (since derivatives up to order 2 are involved).

Systems of ODEs

Any ODE can be transformed into a **system of ODEs of order 1**. It is practical if we have ODEs of order higher than 1.

Examples:

- The following ODE, $\log(x'') - x'^2 + 3x = t$ (order 2) is equivalent to the order 1 system

$$\begin{cases} y = x' \\ \log(y') - y^2 + 3x = t \end{cases}$$

Systems of ODEs

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Examples:

- The system

$$\begin{cases} x'' + xy = z \\ y'z = \cos(tx) \\ z'' = e^{xy} \end{cases}$$

  is equivalent to the order 1 system

$$\begin{cases} u = x' \\ v = z' \\ u' + xy = z \\ y'z = \cos(tx) \\ v' = e^{xy} \end{cases}$$

First order system of ODEs with constant coefficients

Any **first-order explicit linear system of ODEs with constant coefficients** can be written in matrix form.

Example:

$$\begin{cases} x' = 3x - z + e^t \\ y' = -3x - t \\ z' = x + y + z \end{cases} \quad \Leftrightarrow \quad \vec{x}' = A\vec{x} + \vec{b}(t)$$

with

$$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad A = \begin{pmatrix} 3 & 0 & -1 \\ -3 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad \text{and} \quad \vec{b}(t) = \begin{pmatrix} e^t \\ -t \\ 0 \end{pmatrix}.$$

First order system of ODEs with constant coefficients

We are left to study first-order ODEs of the form $\vec{x}'(t) = A\vec{x}(t) + \vec{b}(t)$ where:

- $\vec{x}(t)$ is an unknown vector-valued function of size n,
- $\vec{b}(t)$ is a known vector-valued function of size n,
- A is a known $n \times n$ matrix with constant coefficients.

Preliminaries: eigenvalues and eigenvectors

Definition

Let A be an $n \times n$ matrix.
A non-zero vector $V \in \mathbb{R}^n$ is called an **eigenvector** of A if there exists a scalar $\lambda \in \mathbb{R}$ such that:

$$AV = \lambda V.$$

The scalar λ is called an **eigenvalue** of A corresponding to the eigenvector V.

Remark: We denote $\mathrm{Sp}(A)$ the set of eigenvalues of A.

Procedure to finding eigenvalues and eigenvectors

Let A be an $n \times n$ matrix.

1. First, **find the eigenvalues** λ of A by solving the equation $\det(A - \lambda I) = 0$.
2. Then, for each eigenvalue λ, **find the associated eigenvectors** $V \neq 0$ by solving the system $(A - \lambda I)V = 0$.

Remark: You can verify your work by making sure that $AV = \lambda V$ for each eigenvalue λ and its associated eigenvector V.

Example:

$A = \begin{pmatrix} 5 & -2 \\ -1 & 4 \end{pmatrix}$ has eigenvalues $\lambda_1 = 3$ with eigenvector $V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\lambda_2 = 6$ with eigenvector $V_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. Here, $\mathrm{Sp}(A) = \{3, 6\}$.

First order system of ODEs with constant coefficients

We are left to study first-order ODEs of the form $\vec{x}'(t) = A\vec{x}(t) + \vec{b}(t)$.

- We first consider the **associated homogeneous ODE**: $\vec{x}'(t) = A\vec{x}(t)$.
  We look at solutions of the form $\vec{x}_h(t) = e^{\lambda t}\vec{U}$, where $\vec{U}$ is a constant vector.
  In this case, we have that

$$\vec{x}_h'(t) = \lambda e^{\lambda t}\vec{U} \qquad \text{and} \quad A\vec{x}_h(t) = e^{\lambda t}A\vec{U},$$

  which implies that $A\vec{U} = \lambda\vec{U}$, meaning that λ is an eigenvalue of A with associated eigenvector $\vec{U}$.
  Conversely, it is easy to check that if $A\vec{U} = \lambda\vec{U}$, then $\vec{x}_h(t) = e^{\lambda t}\vec{U}$ is a solution of the ODE.

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Any $\vec{x}_h(t) = e^{\lambda t}\vec{U}$, where λ is an eigenvalue of A and $\vec{U}$ is its associated eigenvector, is a solution of the homogeneous ODE $\vec{x}'(t) = A\vec{x}(t)$.

First order system of ODEs with constant coefficients

- When A is a **diagonalizable** matrix, this implies that we have found all solutions, because it is a known result that the space of solutions is n-dimensional. Hence, in this case, the general solution of $\vec{x}'(t) = A\vec{x}(t)$ is

$$\vec{x}(t) = \alpha_1 e^{\lambda_1 t}\vec{U}_1 + \cdots + \alpha_n e^{\lambda_n t}\vec{U}_n,$$

  where the $\vec{U}_i$ form a basis of eigenvectors of the matrix A, and the λ_i are the associated eigenvalues (possibly complex-valued).

- When A is **not diagonalizable**, we get further solutions for eigenvalues λ having multiplicities using functions of the form $te^{\lambda t}$, $t^2 e^{\lambda t}$, etc.

First order system of ODEs with constant coefficients

- We now consider the **non-homogeneous ODE**: $\vec{x}'(t) = A\vec{x}(t) + \vec{b}(t)$. We want to find a solution using the **integrating factor method**.

First order system of ODEs with constant coefficients

- We now consider the **non-homogeneous ODE**: $\vec{x}'(t) = A\vec{x}(t) + \vec{b}(t)$. We want to find a solution using the **integrating factor method**.
  - We multiply both sides of the equation by e^{-tA}:

$$e^{-tA}\vec{x}'(t) - e^{-tA}A\vec{x}(t) = e^{-tA}\vec{b}(t) \quad \Leftrightarrow \quad \left(e^{-tA}\vec{x}(t)\right)' = e^{-tA}\vec{b}(t)$$

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  - We integrate both sides with respect to t:

$$e^{-tA}\vec{x}(t) = \int e^{-tA}\vec{b}(t)dt + \vec{C} \quad \Leftrightarrow \quad \vec{x}(t) = e^{tA}\vec{C} + e^{tA}\int e^{-tA}\vec{b}(t)dt$$

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  - We integrate both sides with respect to t:

$$e^{-tA}\vec{x}(t) = \int e^{-tA}\vec{b}(t)dt + \vec{C} \quad \Leftrightarrow \quad \vec{x}(t) = e^{tA}\vec{C} + e^{tA}\int e^{-tA}\vec{b}(t)dt$$

Remarks:

- The integration means that each component of the vector $\mathrm{e}^{-tA}\vec{b}(t)$ is integrated separately.
- The constant $\vec{C}$ is a vector of constants, which can be determined using initial conditions as seen in previous lectures.