Practicals on Ordinary Differential Equations

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Why are ODEs important?

ODEs allow us to **model dynamic phenomena**, such as population growth, because they relate a quantity to its rate of change.

Historical perspective: Malthus (1798)
Let α be the birth rate and β be the death rate (both assumed constants), then:

$$x'(t) = \alpha x(t) - \beta x(t) = (\alpha - \beta) x(t)$$

can represent the growth of a population $x(t)$.

Simplistic model for population growth:

$$x'(t) = a x(t), \quad a \text{ constant.}$$

- $x(t)$ represents the population at time t.
- a is the growth rate ($a = \alpha - \beta$): $a > 0$ means the population is growing, $a < 0$ means the population is decreasing and $a = 0$ means the population is constant.

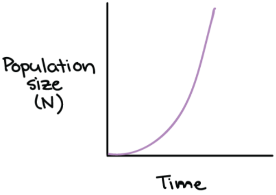
Why are ODEs important?

Simplistic model for population growth: the **exponential model**

$$x'(t) = ax(t), \quad a > 0.$$

- Assumption: The growth rate $x'(t)$ is directly proportional to the current population $x(t)$.
- Implication: Larger populations grow faster $\rightarrow$ **exponential growth**.

Drawback: $x(t) = x(0)e^{at}$ increases without bound, which is unrealistic.



Why are ODEs important?

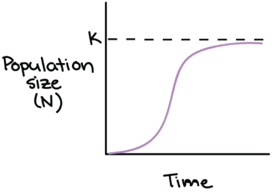
Historical perspective: Verhulst (1845)
Idea: include further assumptions.

- If the population is small, the growth rate remains directly proportional to the size of the population.
- If the population grows too large, however, the growth rate becomes negative.

This lead to introducing the **logistic model**:

$$x'(t) = ax(t)\Big(1 - \frac{x(t)}{K}\Big), \quad a > 0, \quad K > 0$$

where a is the growth rate and K is the carrying capacity (ideal population).



The logistic model

$$x'(t) = ax(t)\Big(1 - \frac{x(t)}{K}\Big), \quad a > 0, \quad K > 0$$

Questions:
Explain how this new model accomodates with the additional assumptions made in the modeling process.

- If the population is small ($x \ll K$), then $1 - \frac{x(t)}{K} \approx 1$. Replacing this in the ODE, we obtain $x'(t) \approx ax(t)$ which is the Malthus model. Therefore, the growth rate remains directly proportional to the size of the population.
- If the population grows too large ($x \gg K$), then $1 - \frac{x(t)}{K} < 0$. Replacing this in the ODE, we obtain $x'(t) < 0$, therefore the growth rate becomes negative.

Remark: Many other differential equations correspond to these assumptions, but the previous choice is presented here for simplicity.

The logistic model

$$x'(t) = ax(t)\Big(1 - \frac{x(t)}{K}\Big), \quad a > 0, \quad K > 0$$

Questions:
Assuming that $K = 1$, solve the equation and interpret the results. What is the qualitative behavior of the solutions?

Explicit computations lead to the solution:

$$x(t) = \frac{x(0)e^{at}}{1 - x(0) + x(0)e^{at}},$$

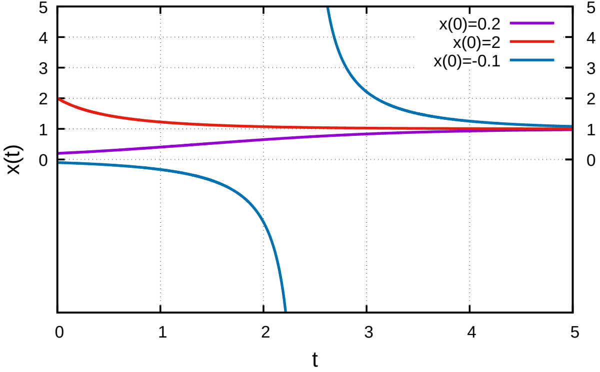
where $x(0)$ is a given initial population.

Equilibrium solutions are given by: $x(t) \equiv 0$ and $x(t) \equiv 1$.

Reminder: Equilibrium solutions are solutions to differential equations where the derivative equals zero along that solution.

The logistic model

Questions:
Assuming that $K = 1$, solve the equation and interpret the results. What is the qualitative behavior of the solutions?



Different solutions for $x(0) > 1$, $0 < x(0) < 1$ and $x(0) < 0$.

Applications

Spruce Budworm Outbreak Model

$$x'(t) = ax(t)\left(1 - \frac{x(t)}{K}\right) - f(x)$$

where $a > 0$ is the rate of population growth, K is the carrying capacity, and f models the mortality of budworms due to predatory birds.

Questions:

- Choice of f?
- Interpretation?
- Solutions?
- Equilibrium points?

Ludwig D., Jones D.D., and Holling C.S. *Qualitative analysis of insect outbreak systems: the spruce budworm and forest.* J. Anim. Ecol., 47:315–332 (1978).

Applications

Spruce Budworm Outbreak Model

$$x'(t) = ax(t)\left(1 - \frac{x(t)}{K}\right) - f(x)$$

Questions:
Choice of f?

$$f(x) = \frac{x^2}{1+x^2}.$$

- $f(x) \to 1$ when $x \to +\infty$: predation is limited, and an outbreak cannot be avoided.
- $f(x) \to 0$ when $x \to 0$: when the budworm population is small, birds consume other prey.
- Predation increases in a steep manner.

Ludwig D., Jones D.D., and Holling C.S. *Qualitative analysis of insect outbreak systems: the spruce budworm and forest.* J. Anim. Ecol., 47:315–332 (1978).

Applications

Spruce Budworm Outbreak Model

$$x'(t) = ax(t)\left(1 - \frac{x(t)}{K}\right) - f(x)$$

Questions:
Solutions? Equilibrium points?

Difficulties:

- No analytical solution is available.
- To find equilibrium points, we need to solve:

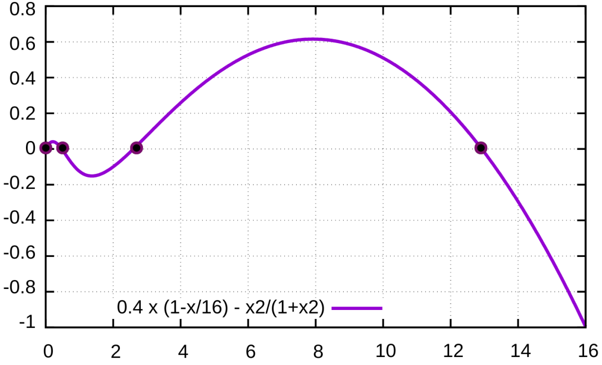
$$x'(t) = 0 \quad \Leftrightarrow \quad ax(t)\left(1 - \frac{x(t)}{K}\right) = \frac{x^2(t)}{1 + x^2(t)}.$$

- **Remark:** $x(t) \equiv 0$ is always a solution, but other equilibrium points may exist.

Applications

Questions:

Solutions? Equilibrium points?



In the case $a = 0.4$ and $K = 16$, the equation has four equilibrium points.
$\Rightarrow$ We need an **alternative approach** to solve the problem.