Introduction to Ordinary Differential Equations (ODEs)

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Basic notions and examples

Main concept: the derivative of a function.

Definition (analytical): The derivative of a function $y = f(x)$ at a point x_0, denoted $\frac{dy}{dx}(x_0)$ or $f'(x_0)$, is defined as:

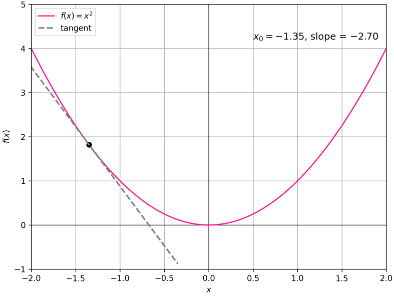
$$\frac{dy}{dx}(x_0) = f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

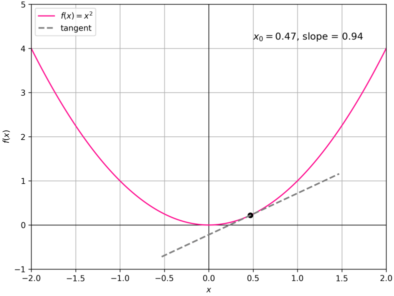
$$\uparrow$$

instantaneous rate of change of f at x_0,

assuming that the limit exists and is finite.

Definition (geometrical): The derivative of a function, geometrically, represents the slope of the tangent line to the graph of the function at a specific point. Essentially, it captures the instantaneous rate of change of the function at that point.





Basic notions and examples

Ordinary Differential Equations (ODEs): equations involving an unknown function $x(t)$, its derivative $x'(t)$, and possibly its higher-order derivatives $x''(t)$, $x^{(3)}(t)$, *etc.*, up to a given order $x^{(n)}(t)$, which are supposed to be satisfied for each t.

Examples:

- $x'(t) + x(t) = 0$,
- $x'(t) = tx(t) + t^2$,
- $x''(t) = \cos(x'(t))\, x(t)$.

ODEs are called "ordinary" to distinguish them from **Partial Differential Equations** (PDEs), which involve partial derivatives of functions of several variables.

Basic notions and examples

Some remarks about **notation**:

- Derivatives are sometimes written as $\frac{dx}{dt}(t)$, $\frac{d^2x}{dt^2}(t)$ (the standard notation), or as $x'(t)$, $x''(t)$, or $\dot{x}(t)$, $\ddot{x}(t)$.
- In this course, we will use the notation $x(t)$, with x being the unknown function and t the variable. However, other notations can be found in different textbooks: $f(t)$, $f(x)$, $y(t)$, $y(x)$, etc.
- Sometimes we drop the "(t)" in the equations to simplify notation. For example, the three previous ODEs could also be written as: $x' + x = 0$, $x' = tx + t^2$, $x'' = \cos(x')x$.

Some vocabulary for ODEs

Orders of ODEs

The **order** of an equation is the order of the highest derivative that appears. For example, if the highest derivative that appears is the first derivative, the equation is of first order; if the highest derivative that appears is the second derivative, the equation is of second order.

Examples:

- $x'(t) + x(t) = 0$ is an ODE of order 1,
- $x'(t) = tx(t) + t^2$ is an ODE of order 1,
- $x''(t) = \cos(x'(t))\, x(t)$ is an ODE of order 2.

Some vocabulary for ODEs

Autonomous ODEs

An ODE is **autonomous** when the variable t does not appear explicitly in the equation, except through the function $x(t)$ and its derivatives.

Examples:

- $x''(t)^2 + x'(t) + x(t) = 0$ is an autonomous ODE,
- $x''(t) = t + \cos(x(t))$ is not autonomous.

Some vocabulary for ODEs

Linear ODEs

A **linear** ODE is an ODE of the form

$$a_0(t)x(t) + a_1(t)x'(t) + \cdots + a_n(t)x^{(n)}(t) = b(t),$$

where the $a_i(t)$ and $b(t)$ are known functions.

- A linear ODE is **homogeneous** if and only if $b(t) = 0$.
- A linear ODE has **constant coefficients** if the $a_i(t)$ are constants.

Examples:

- $tx + e^t x' = 2t$ is a linear ODE,
- $x'' - x' + 4x = 5\cos(t)$ is a linear ODE with constant coefficients,
- $x' - tx = 0$ is a homogeneous linear ODE.

Explicit verification: The Guess and Check method

For a given ODE and a candidate function, it is easy to check whether this function is actually a solution or not.

Examples:

- $x(t) = e^{-t}$ is a solution of $x' + x = 0$.
- $x(t) = \cos(t)$ is a solution of $x'' + x = 0$.

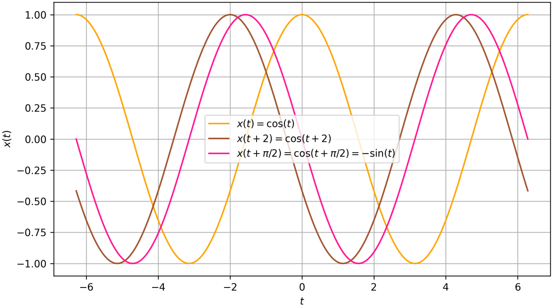
Let's check this together!

Remark: Other solutions may exist!
For example, $x(t) = \sin(t)$ is also a solution of $x'' + x = 0$, since $x'(t) = \cos(t)$ and $x''(t) = -\sin(t) = -x(t)$.

Explicit verification: The Guess and Check method

Remark: If $x(t)$ is a solution of an **autonomous ODE**, then any function of the form $x(t+\lambda)$, where λ is a constant, is also a solution.

Example: $x(t) = \cos(t)$, $x(t) = \cos(t+2)$, and $x(t) = \cos(t+\pi/2) = -\sin(t)$ are all solutions of the ODE $x'' + x = 0$.
Graphically, this means that by translating the graph of a solution along the horizontal axis, we obtain the graphs of other solutions of the equation.



A simple example

For the following ODE:

$$x'(t) = ax(t),$$

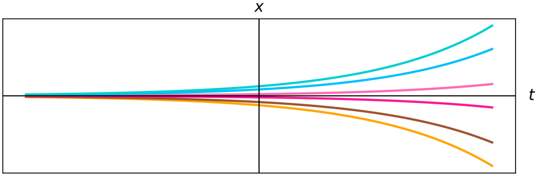
with $a \in \mathbb{R}$ (a is a given parameter), $x(t) = ke^{at}$ satisfies the equation for any real number k, and there are no other solutions.

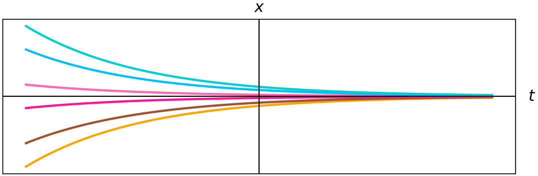
Remarks:

- Given the initial condition $x(t_0) = u_0$, we obtain $k = u_0 e^{-at_0}$.
- If $k = 0$, then the constant solution $x(t) \equiv 0$ is obtained. A constant solution of this type is called an **equilibrium solution** or an **equilibrium point** for the equation.
- The behavior of the solutions differs when $a > 0$, $a = 0$, or $a < 0$.

A (not so) simple example

The solution graphs for $a > 0$ (top) and $a < 0$ (bottom) are shown below. Each graph represents a particular solution for a different initial condition.





Questions

- Can we always prove **existence** and **uniqueness** of solutions?

  Difficulties: Some nonlinear systems have **no solutions** whatsoever for a given initial condition, while others may have **infinitely many different solutions**. In some cases, solutions may not be defined for all time or may tend to ∞ in finite time.

- Can we always find an **explicit** solution?

  Difficulties: Most nonlinear systems of differential equations are impossible to solve analytically.

- What happens if we **vary the initial condition** of a system ever so slightly? Does the corresponding solution vary continuously?

Theoretical results

Let:

- $\mathcal{O} \subset \mathbb{R} \times \mathbb{R}^n$ be an open set, and $(t_0, X_0) \in \mathcal{O}$.
- $F : \mathcal{O} \to \mathbb{R}^n$ be a $\mathcal{C}^1$ (continuously differentiable) function in X, and continuous (possibly only) in t.

Consider the **nonautonomous differential equation:**

$$X'(t) = F(t, X), \quad X(t_0) = X_0.$$

A **solution** of this system is a differentiable curve $X(t)$ in $\mathbb{R}^n$ defined on some interval J, having the following properties:

- $t_0 \in J$ and $X(t_0) = X_0$,
- $(t, X(t)) \in \mathcal{O}$ and $X'(t) = F(t, X(t))$ for all $t \in J$.

Theoretical results

Fundamental local theorem for nonautonomous equations:

Theorem

Let $\mathcal{O} \subset \mathbb{R} \times \mathbb{R}^n$ be an open set, and let $F : \mathcal{O} \to \mathbb{R}^n$ be a $\mathcal{C}^1$ function in X and continuous in t.
If $(t_0, X_0) \in \mathcal{O}$, there exists an open interval J containing t_0 and a unique solution of $X'(t) = F(t, X)$ defined on J and satisfying $X(t_0) = X_0$.

- **Pros:** This result ensures existence and uniqueness of solutions, provided certain conditions are imposed on the function F.
- **Cons:** It does not provide an explicit construction of the solution.

Solving first-order linear ODEs with constant coefficients

General form:

$$x'(t) = ax(t) + b(t),$$

where $a \in \mathbb{R}$ is constant.

Step 1: Solve the homogeneous equation

$$x'(t) = ax(t) \quad \Rightarrow \quad x_h(t) = Ce^{at}.$$

Step 2: Variation of constants: Replace C by a function $C(t)$:

$$x_p(t) = C(t)e^{at}.$$

Step 3: Find $C(t)$: Differentiate $x_p(t)$ and substitute into the ODE:

$$C'(t)e^{at} = b(t) \quad \Rightarrow \quad C(t) = \int b(t)e^{-at}\,dt + s, \quad s \text{ integration constant.}$$

General solution:

$$x(t) = x_h(t) + x_p(t) = Ke^{at} + e^{at}\int b(t)e^{-at}\,dt, \quad \text{with } K = C + s.$$

Solving first-order linear ODEs with constant coefficients

General solution formula: For the ODE $x'(t) = ax(t) + b(t)$, the solution is

$$x(t_1) = e^{a(t_1 - t_0)} x(t_0) + \int_{t_0}^{t_1} b(t)\, e^{-a(t-t_1)}\, dt,$$

where $t_1 \in \mathbb{R}$ and $x(t_0)$ is the initial condition.

Tricks to solve this type of ODE:

- Rewrite the ODE as $x'(t) = ax(t) + b(t)$ and identify a and $b(t)$.
- Use the formula with $t_0 = 0$ if possible.
- Compute the integral and give the explicit solution at t_1.

Why does this work? $\rightarrow$ multiplication by e^{-at}.

Examples of first-order linear ODEs

Solve the following ODEs with initial condition $x(t_0) = x(0) = x_0$:

- $x'(t) - t^2 = 0$,
- $x'(t) - 3x(t) = 4$,
- $x'(t) - e^t = -2x(t)$,
- $x'(t) = x(t) + t$,
- $x'(t) + x(t) = \sin(t)$.

Examples of first-order linear ODEs - correction

- $x'(t) - t^2 = 0$

  Solution: $x(t) = x_0 + \frac{t^3}{3}$

- $x'(t) - 3x(t) = 4$

  Solution: $x(t) = \left(x_0 + \frac{4}{3}\right)e^{3t} - \frac{4}{3}$

- $x'(t) - e^t = -2x(t)$

  Solution: $x(t) = \left(x_0 - \frac{1}{3}\right)e^{-2t} + \frac{1}{3}e^t$

Examples of first-order linear ODEs - correction

- $x'(t) = x(t) + t$

  Solution: $x(t) = (x_0 + 1)e^t - t - 1$

- $x'(t) + x(t) = \sin(t)$

  Solution: $x(t) = \left(x_0 + \frac{1}{2}\right)e^{-t} + \frac{1}{2}\big(\sin(t) - \cos(t)\big)$